JOURNAL OF COMBINATORIAL THEORY (B) 13, 69–70 (1972)

Monochromatic Paths in Edge-Colored Graphs

V. CHVÁTAL

Department of Mathematics, McGill University, Montreal 101, Quebec, Canada

Communicated by W. T. Tutte

Received November 5, 1971

A theorem due to Busolini [1], which appeared in this journal, can be shortened and strengthened as follows:

THEOREM. *Let $G = (V, E)$ be a finite directed graph without loops where V is the set of vertices of G and $E = E_1 \cup E_2 \cup \cdots \cup E_k$ is the set of edges of G. Let*

$$\chi(G) > n_1 n_2 \cdots n_k, \tag{1}$$

where $n_1, n_2, \ldots, n_k$ are positive integers and $\chi(G)$ is the chromatic number of G. Then there is an integer j with $1 \leqslant j \leqslant k$ such that $G_j = (V, E_j)$ contains a (simple) directed path with n_j edges.

Proof. One has $\chi(G) \leqslant \chi(G_1)\, \chi(G_2) \cdots \chi(G_k)$. Indeed, if $f_j : V \to X_j$ is a coloring of G_j (i.e., $(u, v) \in E_j \Rightarrow f_j(u) \neq f_j(v)$) for each j then $f: V \to X_1 \times X_2 \times \cdots \times X_k$ defined by $f(u) = (f_1(u), f_2(u), \ldots, f_k(u))$ is a coloring of G. Hence $\chi(G_j) > n_j$ for some j. By a theorem of Gallai [3], $\chi(G_j) > n_j$ implies that G_j contains a directed path of n_j edges. Q.E.D.

Now, if (1) is replaced by the (stronger) assumption $|V| > \alpha n_1 n_2 \cdots n_k$ (where α is the maximum cardinality of any independent set of vertices in G), then one obtains a stronger version of Busolini's theorem. Our theorem is contained implicitly in Theorem 1 of [2]; see also paragraph 8 of [4]. Related results were also obtained by V. Rödl (Charles University, Prague).

REFERENCES

1. D. T. BUSOLINI, Monochromatic paths and circuits in edge-colored graphs, *J. Combinatorial Theory* **10** (1971), 299–300.
2. V. CHVÁTAL AND J. KOMLÓS, Some combinatorial theorems on monotonicity, *Canad. Math. Bull.* **14** (1971), 151–157.

3. T. GALLAI, On directed paths and circuits, "Theory of Graphs" (P. Erdös and G. Katona, eds.), Academic Press, New York, 1968, pp. 115–119.
4. R. RADO, Theorems on the colouring of the edges of a graph, Proceedings of the Second Chapel Hill Conference on Combinatorial Mathematics and Its Applications, University of North Carolina, 1970, pp. 385–390.